# A relation between definable $G$ vector bundles and definable fiber bundles (Model theoretic aspects of the notion of independence and dimension)



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# A relation between definable $G$ vector bundles and definable fiber bundles

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## 1 Introduction

Let $G$ be a compact Lie group. It is well-known that the set of isomorphism classes of $G$ vector bundle over a $G$ space space with free action corresponds bijectively to the set of isomorphism classes of vector bundles over the orbit space [1].

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field $R$. Everything is considered in $\mathcal{N}$ and the term "definable" is used throughout in the sense of "definable with parameters in $\mathcal{N}$", each definable map is assumed to be continuous.

General references on o-minimal structures are [2], [3], also see [6].

In this paper we prove that the set of isomorphism classes of definable $G$ vector bundles over a definable $G$ set $X$ is in one–to-one correspondence to that of definable vector bundles over a definable set $X/G$ when the action on $X$ is free.

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## 2 Our result

A field $(R, +, \cdot, <)$ with a dense linear order $<$ without endpoints is an *ordered field* if it satisfies the following two conditions.

(1) For any $x, y, z \in R$, if $x < y$, then $x + z < y + z$.

(2) For any $x, y, z \in R$, if $x < y$ and $z > 0$, then $xz < yz$.

An ordered field $(R, +, \cdot, <)$ is a *real field* if for any $y_1, \dots, y_m \in R$, $y_1^2 + \cdots + y_m^2 = 0 \Rightarrow y_1 = \cdots = y_m = 0$.

A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.

(1) For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.

(2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal* (*order minimal*) if every definable set of $R$ is a finite union of open intervals and points, where open interval means $(a, b), -\infty \leq a < b \leq \infty$.

If $(R, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.

The topology of $R$ is the interval topology and the topology of $R^n$ is the product topology.

Let $X \subset R^n$ and $Y \subset R^m$ be definable sets. A continuous map $f : X \to Y$ is *definable* if the graph of $f$ $(\subset X \times Y \subset R^n \times R^m)$ is a definable set. A definable map $f : X \to Y$ is a *definablehomeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

A group $G$ is a *definable group* if $G$ is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable.

Let $G$ be a definable group. A pair $(X, \phi)$ consisting a definable set $X$ and a $G$ action $\phi : G \times X \to X$ is a *definable $G$ set* if $\phi$ is definable. We simply write $X$ instead of $(X, \phi)$ and $gx$ instead of $\phi(g, x)$.

A definable map $f : X \to Y$ between definable $G$ sets is a *definable $G$ map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$. A definable $G$ map is a *definable $G$ homeomorphism* if it is a homeomorphism.

A definable set $X$ is *definably compact* if for every $a, b \in R \cup \{\infty\} \cup \{-\infty\}$ with $a < b$ and for every definable map $f : (a, b) \to X$, $\lim_{x \to a+0} f(x)$ and $\lim_{x \to b-0} f(x)$ exist in $X$. If $R = \mathbb{R}$, then for any definable subset $X$ of $\mathbb{R}^n$,

$X$ is compact if and only if it is definably compact. In general a definably compact set is not necessarily compact. For example, if $R = \mathbb{R}_{alg}$, then $[0,1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1\}$ is definably compact but not compact.

**Theorem 2.1** ([5]). *Let $X$ be a definable subset of $R^n$. Then $X$ is definably compact if and only if $X$ is closed and bounded.*

**Theorem 2.2** (Existence of definable quotient ([2])). *Let $G$ be a definably compact definable group and $X$ a definable $G$ set. Then the orbit space $X/G$ exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.*

Let $X$ be a definable $G$ set. The action on $X$ is $free$ if for any $x$ in $X$, the isotropy subgroup $G_x = \{g \in G | gx = x\}$ of $x$ is the trivial group.

**Definition 2.3.** *A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a $definable\ fiber\ bundle$ over $X$ with fiber $F$ and structure group $K$ if the following two conditions are satisfied:*

*(1) The total space $E$ is a definable space, the base space $X$ is a definable set, the structure group $K$ is a definable group, the fiber $F$ is a definable set with an effective definable $K$ action, and the projection $p : E \to X$ is a definable map.*

*(2) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of $\eta$ such that each $U_i$ is a definable open subset of $X$, $\{U_i\}_i$ is a finite open covering of $X$. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where $\pi_i$ stands for the projection $U_i \times F \to F$. For any $i$ and $j$ with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations $definable$.*

Definable fiber bundles with compatible definable local trivializations are identified.

Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a $definable\ fiber\ bundle$ $morphism$ if the following two conditions are satisfied:

(1) The map $\overline{f}$ covers a definable map, namely there exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(2) For any $i, j$ such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in $K$, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

We say that a bijective definable fiber bundle morphism $\overline{f} : E \to E'$ is a *definable fiber bundle equivalence* if it covers a definable homeomorphism $f : X \to X'$ and $(\overline{f})^{-1} : E' \to E$ is a definable fiber bundle morphism covering $f^{-1} : X' \to X$. A definable fiber bundle equivalence $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$ and $f = id_X$.

A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any $i$, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the $K$ action on $F$ is defined by the multiplication of $K$. We write $(E, p, X, K)$ for $(E, p, X, F, K)$.

**Definition 2.4.** *(1) A definable fiber bundle $\eta = (E, p, X, F, K)$ is a definable vector bundle if $F = R^n, K = GL(n, R)$.*

*(2) Let $G$ be a definable group. A definable vector bundle $\eta = (E, p, X)$ is a definable $G$ vector bundle if $E, X$ are definable $G$ sets, $p : E \to X$ is a definable $G$ map, and $G$ acts on $E$ by definable vector bundle isomorphism.*

Our result is the following.

**Theorem 2.5** ([4])**.** *Let $G$ be a definably compact definable group and $X$ a definable $G$ set. If $G$ acts on $X$ freely, then the set of isomorphism classes of definable $G$ vector bundles over $X$ corresponds bijectively to the set of isomorphism classes of definable vector bundles over $X/G$.*

# References


[1] M. F. Atiyah, *K-theory*, Benjamin, 1967.

[2] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).

[3] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.

[4] T. Kawakami, *Definable G vector bundles over a definable G set with free action*, to appear.

[5] Y. Peterzil and C. Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, J. London Math. Soc. **59** (1999), 769–786.

[6] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics **150**, Birkhäuser, Boston, 1997.